4.2 ⋆⋆ **(a)**

$$W = (\int_O^Q + \int_Q^P)\mathbf{F}\cdot d\mathbf{r} = \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,0)\,dy = \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{1}{3} + 1 = \frac{4}{3}.$$

(b) On this path $y = x^2$, so

$$W = \int_O^P F_x\,dx + \int_O^P F_y\,dy = \int_0^1 x^2\,dx + \int_0^1 (2x^3)(2x\,dx) = \frac{1}{3} + \frac{4}{5} = \frac{17}{15}.$$

(c) On this path $x = t^3$ and $y = t^2$, so

$$W = \int_O^P F_x\,dx + \int_O^P F_y\,dy = \int_0^1 t^6(3t^2\,dt) + \int_0^1 (2t^5)(2t\,dt) = 3 \times \frac{1}{9} + 4 \times \frac{1}{7} = \frac{19}{21}.$$

4.4 ⋆⋆ **(a)** As in Problem 3.25, conservation of angular momentum implies that $mr^2\omega = mr_o^2\omega_o$, so $\omega = (r_o/r)^2\omega_o$.

(b) The tension force, which I must supply, is what keeps the particle in its circular path with centripetal acceleration $a_r = -\omega^2 r$. (This is where we must assume that I pull the string slowly — otherwise, $a_r = \ddot{r} - \omega^2 r$.) Thus the force which I exert is

$$F(r) = m\omega^2 r = m\left[\left(\frac{r_o}{r}\right)^2\omega_o\right]^2 r = m\omega_o^2 r_o^4 \frac{1}{r^3}$$

where I used the result of part (a) for the second equality. The work I do is (Remember the distance I pull the string in any small displacement is $-dr$.)

$$W = \int_{r_o}^{r} F(r')(-dr') = -m\omega_o^2 r_o^4 \int_{r_o}^{r} \frac{dr'}{r'^3} = \frac{1}{2}m\omega_o^2 r_o^4\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right).$$

(c) The particle's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$. Thus, with the result of part (a) for ω,

$$\Delta T = \frac{1}{2}m(r^2\omega^2 - r_o^2\omega_o^2) = \frac{1}{2}m\omega_o^2\left(\frac{r_o^4}{r^2} - r_o^2\right)$$

which is the same as the work done in part (b), as it has to be.

4.8 ⋆⋆ We'll measure the puck's position by the angle θ it subtends at the sphere's center O (measured down from the top). The puck's PE (defined as zero at the level of O) is $U(\theta) = mgR\cos\theta$, and its total energy is $E = U(0) = mgR$. By conservation of energy,

$$T = \tfrac{1}{2}mv^2 = E - U = mgR(1 - \cos\theta). \tag{i}$$

As long as the puck remains in contact with the sphere, the radial component of Newton's second law reads $N - mg\cos\theta = -mv^2/R$, where N denotes the normal force of the sphere on the puck. Substituting from Eq (i) for mv^2 we find

$$N = mg(3\cos\theta - 2).$$

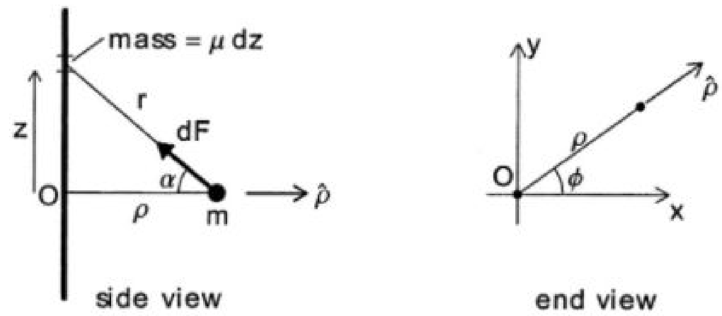
As long as N is positive the puck remains on the sphere. Since the sphere cannot exert a negative normal force, once the predicted value of N becomes negative, the puck must have left the sphere. Therefore it leaves the sphere when $N = 0$ or $\theta = \arccos(2/3) = 48.2^\circ$ and the height below the top is $R/3$.

4.12 ⋆ **(a)** $\nabla f = \hat{\mathbf{x}}\,2x + \hat{\mathbf{z}}\,3z^2$. **(b)** $\nabla f = k\,\hat{\mathbf{y}}$. **(c)** $\nabla f = \hat{\mathbf{r}}$. **(d)** $\nabla f = -\hat{\mathbf{r}}/r^2$

4.18 ⋆⋆ **(a)** According to (4.35), the change in $f(\mathbf{r})$ resulting from any small displacement $d\mathbf{r}$ is $df = \nabla f \cdot d\mathbf{r}$. If, in particular, we consider any infinitesimal displacement $d\mathbf{r}$ in a surface of constant f, then df will be zero. This implies that $\nabla f \cdot d\mathbf{r} = \mathbf{0}$, that is, ∇f is perpendicular to the surface of constant f.

(b) Consider a displacement $d\mathbf{r} = \epsilon\mathbf{u}$ with fixed magnitude ϵ but variable direction $\mathbf{u}$. Our job is to find the direction of $\mathbf{u}$ for which the corresponding change df is largest. Since $df = \nabla f \cdot d\mathbf{r} = \epsilon\nabla f \cdot \mathbf{u} = \epsilon|\nabla f|\cos\theta$, where θ is the angle between ∇f and $\mathbf{u}$, we see that df is maximum if $\theta = 0$, or ∇f and $\mathbf{u}$ are parallel. That is, the direction of ∇f is the direction in which f increases most rapidly.

4.24 ⋆⋆⋆ **(a)** Consider first the force on m due to a short segment dz of the rod at a height z above m. This force has magnitude $dF = Gm\mu\, dz/r^2$ in the direction shown in the left picture, where r is the distance from the element dz to m. To find the total force we must integrate this from $z = -\infty$ to ∞. When we do this, the z components F_z from points z and $-z$ will cancel. Since the component into the page is clearly zero, we have only to worry



about the component in the direction of $\hat{\boldsymbol{\rho}}$ (the unit vector in the ρ direction, pointing away from the z axis):

$$dF_\rho = -Gm\mu\cos\alpha\frac{dz}{r^2} = -Gm\mu\rho\frac{dz}{r^3}$$

where the last expression follows because $\cos\alpha = \rho/r$. Thus the net force has ρ component

$$F_\rho = -Gm\mu\rho\int_{-\infty}^{\infty}\frac{dz}{r^3} = -Gm\mu\rho\int_{-\infty}^{\infty}\frac{dz}{(z^2+\rho^2)^{3/2}} = -\frac{Gm\mu}{\rho}\int_{-\pi/2}^{\pi/2}\cos\alpha\, d\alpha$$

where the last form results from the substitution $z/\rho = \tan\alpha$. The final integral is just 2, and we conclude that

$$\mathbf{F} = -\frac{2Gm\mu}{\rho}\hat{\boldsymbol{\rho}}\,. \qquad \text{(ii)}$$

(b) The unit vector $\hat{\boldsymbol{\rho}}$ lies in the xy plane. If we denote its polar angle by ϕ as in the right picture, then

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$$

where $\cos\phi = x/\rho$ and $\sin\phi = y/\rho$. Substituting into Eq. (ii), we find

$$\mathbf{F} = -\frac{2Gm\mu}{\rho^2}(\hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,y + \hat{\mathbf{z}}\,0)$$

where $\rho = \sqrt{x^2+y^2}$. It is now a straightforward matter to evaluate the components of $\boldsymbol{\nabla}\times\mathbf{F}$. For instance, $(\boldsymbol{\nabla}\times\mathbf{F})_x = \partial_y F_z - \partial_z F_y$ where I have introduced the abbreviation ∂_x for $\partial/\partial x$ and so on. Since $F_z = 0$ and F_y is independent of z, it follows that $(\boldsymbol{\nabla}\times\mathbf{F})_x = 0$. The y component works in exactly the same way, and

$$(\boldsymbol{\nabla}\times\mathbf{F})_z = \partial_x F_y - \partial_y F_x = -2Gm\mu\left(y\,\partial_x\rho^{-2} - x\,\partial_y\rho^{-2}\right). \qquad \text{(iii)}$$

Now, it is a simple matter to check that $\partial_x\rho^{-2} = -2x\rho^{-4}$ and likewise $\partial_y\rho^{-2} = -2y\rho^{-4}$, so the two terms on the right of Eq.(iii) cancel exactly. Thus all three components of $\boldsymbol{\nabla}\times\mathbf{F}$ are zero, and $\mathbf{F}$ is conservative.

(c) From Eq.(ii), we see that $\mathbf{F}$ is especially simple in cylindrical polar coordinates. Specifically $F_\rho = -2Gm\mu/\rho$, which is independent of ϕ and z, while the other two components are zero, $F_\phi = F_z = 0$. Sustituting into the expression inside the back cover for $\boldsymbol{\nabla}\times\mathbf{F}$ in cylindrical polars, we see immediately the $\boldsymbol{\nabla}\times\mathbf{F} = 0$.

(d) The potential energy $U(\mathbf{r})$ is given by the integral $-\int \mathbf{F}\cdot d\mathbf{r}$ taken from any chosen reference point $\mathbf{r}_o$ to the point of interest $\mathbf{r}$. Since the integral is independent of path, we can choose any convenient path. One such choice is given in cylindrical polar coordinates as follows: Let the reference point $\mathbf{r}_o$ be given by coordinates (ρ_o, ϕ_o, z_o) and $\mathbf{r}$ by (ρ, ϕ, z). Now define the path in three stages:

1. Go from $\mathbf{r}_o$ parallel to the z axis until you reach the desired final value z.
2. Next move in a circle of constant ρ and z until you reach the desired final value of ϕ.
3. Finally go radially out in the direction of $\hat{\boldsymbol{\rho}}$ to the final value of ρ.

In the first two legs of this journey the force does no work. In the final leg, $\mathbf{F}$ and $d\mathbf{r}$ point in the $\hat{\boldsymbol{\rho}}$ direction, and the work integral is easily written as an integral over ρ to give

$$U(\mathbf{r}) = -\int_{\rho_o}^{\rho} \left(-\frac{2Gm\mu}{\rho'}\right) d\rho' = 2Gm\mu \ln(\rho/\rho_o).$$

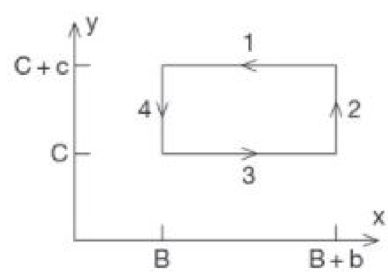
4.25 ⋆⋆⋆ **(a)** Let 1 and 2 denote any two points and Γ_a and Γ_b be any two paths leading from point 1 to point 2. Next let Γ be the closed path that starts at point 1, goes to point 2 via Γ_a, and then returns to point 1 tracing the path Γ_b backwards. Obviously

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_{\Gamma_a} \mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_b} \mathbf{F}\cdot d\mathbf{r}$$

The work integral is path-independent if and only if the right side is zero for any two paths joining any two points 1 and 2, and the left side is zero if and only if the work integral is zero around any closed path. Therefore the two statements are equivalent.

(b) If we accept Stokes's theorem, $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int (\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA$, then obviously $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ if $\boldsymbol{\nabla}\times\mathbf{F} = 0$ everywhere.

(c) The integral going around the closed path Γ can be divided into four integrals, each along one of the straight paths labelled 1, 2, 3, and 4 in the picture.



$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \left(\int_1 + \int_2 + \int_3 + \int_4\right) \mathbf{F}\cdot d\mathbf{r}$$

Now

$$\int_1 + \int_3 = -\int_B^{B+b} F_x(x, C+c, z)dx + \int_B^{B+b} F_x(x, C, z)dx$$

and

$$F_x(x, C+c, z)dx - F_x(x, C, z) = \int_C^{C+c} dy\, \frac{\partial F_x(x,y,z)}{\partial y}.$$

Combining the last two results, we find that

$$\int_1 + \int_3 = -\int_B^{B+b} dx \int_C^{C+c} dy\, \frac{\partial F_x(x,y,z)}{\partial y} = -\int \frac{\partial F_x}{\partial y} dA$$

where the final integral is a two-dimensional integral over the whole rectangle. There is a similar expression (without a minus sign) for $\int_2 + \int_4$, and, combining these two, we conclude that

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dA = \int (\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\, dA.$$

4.28 ⋆⋆ **(a)** Since $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, it follows that $\dot{x}(x) = \sqrt{2/m}\sqrt{E - \frac{1}{2}kx^2}$.

(b) At the end point $x = A$, we know that $T = 0$, so $E = \frac{1}{2}kA^2$. Substituting into the result of part (a), we find $\dot{x}(x) = \omega\sqrt{A^2 - x^2}$, where I have defined $\omega = \sqrt{k/m}$. From (4.58) we find

$$t = \int_0^x \frac{dx'}{\dot{x}(x')} = \frac{1}{\omega}\int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}}.$$

The integral can be evaluated with the substitution $x' = A\sin\theta$ and gives $\arcsin(x/A)$. So $t = (1/\omega)\arcsin(x/A)$.

(c) Solving for x we find $x(t) = A\sin\omega t$. This shows that x is a sinusoidal function of t, which is the definition of simple harmonic motion. In particular, $x(t)$ repeats itself after a time t such that $\omega t = 2\pi$, or $t = 2\pi/\omega = 2\pi\sqrt{m/k}$.

4.30 ⋆ **(a)** As the toy tips, the hemisphere rolls and its center O remains at a fixed height. On the other hand the height of the CM above O changes from $h - R$ to $(h - R)\cos\theta$. Therefore, the PE of the toy is now $U(\theta) = mg[R + (h - R)\cos\theta]$.

(b) Since $dU/d\theta = -mg(h - R)\sin\theta$, which vanishes at $\theta = 0$, we see that the upright position is an equilibrium, as expected. Next, $d^2U/d\theta^2 = -mg(h - R)\cos\theta = mg(R - h)$ at $\theta = 0$. Thus the equilibrium is stable if and only if $R > h$. [If $R = h$, then $U(\theta) = mgR =$ const, and the equilibrium is neutral.]

4.34 ⋆⋆ **(a)** The distance of the mass m below the support is $l\cos\phi$. Therefore, its height measured up from the equilibrium position is $l - l\cos\phi = l(1 - \cos\phi)$ and its PE is $U = mgl(1 - \cos\phi)$. The total energy is $E = \frac{1}{2}ml^2\dot{\phi}^2 + mgl(1 - \cos\phi)$.

(b) The equation $dE/dt = 0$ reads $ml^2\dot{\phi}\ddot{\phi} + mgl\dot{\phi}\sin\phi = 0$ or $ml^2\ddot{\phi} = -mgl\sin\phi$. That is, $I\alpha = \Gamma$.

(c) Provided ϕ remains small, the equation of motion is well-approximated by $l\ddot{\phi} = -g\phi$, whose solution is $\phi = A\cos(\omega t) + B\sin(\omega t)$, where $\omega = \sqrt{g/l}$. This has period $\tau_o = 2\pi\sqrt{l/g}$.

4.36 ⋆⋆ **(a)** It is easy to see that $h = b/\tan\theta$ and $H = l - b/\sin\theta$. Thus

$$U = -mgh - MgH = gb\left(\frac{M}{\sin\theta} - \frac{m}{\tan\theta}\right) = \frac{gb}{\sin\theta}(M - m\cos\theta)$$

where, in the third expression, I dropped an uninteresting constant.

(b) As you can check, the derivative of U is $dU/d\theta = gb(m - M\cos\theta)/\sin^2\theta$. If $m > M$, this never vanishes and there are no equilibrium points. If $m = M$, it vanishes at $\theta = 0$ which is impossible (unless the string is infinitely long). If $m < M$, there is an equilibrium point at $\theta_o = \arccos(m/M)$. Since $\cos\theta$ decreases as θ increases, the factor $(m - M\cos\theta)$ is negative for $\theta < \theta_o$ and positive for $\theta > \theta_o$. Therefore, $U(\theta)$ has a minimum at θ_o and the equilibrium is stable.

4.44 ⋆⋆ Since $\mathbf{F} = f(r)\hat{\mathbf{r}}$, the work done going radially out from A to C is $W_{AC} = \int_A^C \mathbf{F}\cdot d\mathbf{r} = \int_{r_A}^{r_B} f(r)dr$. The same argument applies to W_{DB}, so $W_{AC} = W_{DB}$. On the other hand, on the paths CB and AD, $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, so $W_{CB} = W_{AD} = 0$. Therefore

$$W_{ACB} = W_{AC} + W_{CB} = W_{AD} + W_{DB} = W_{ADB}$$

4.48 ⋆ Let the initial speed of particle 1 be v_1 and the final speed of the composite be v'. Then, conservation of momentum says that $m_1 v_1 = (m_1 + m_2)v'$. Therefore the initial and final KEs are $T = \frac{1}{2}m_1 v_1^2$ and $T' = \frac{1}{2}(m_1 + m_2)v'^2 = \frac{1}{2}m_1^2 v_1^2/(m_1 + m_2)$, and the fractional loss of KE is

$$\frac{T - T'}{T} = \frac{m_1(m_1 + m_2) - m_1^2}{m_1(m_1 + m_2)} = \frac{m_2}{m_1 + m_2}\,.$$

If $m_1 \ll m_2$, almost all the initial KE is lost; if $m_2 \ll m_1$, almost none of the initial KE is lost.